Integrals over Products of Distributions and Coordinate Independence of Zero-Temperature Path Integrals

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Abstract. – In perturbative calculations of quantum-statistical zero-temperature path integrals in curvilinear coordinates one encounters Feynman diagrams involving multiple temporal integrals over products of distributions, which are mathematically undefined. In addition, there are terms proportional to powers of Dirac δ-functions at the origin coming from the measure of path integration. We give simple rules for integrating products of distributions in such a way that the results ensure coordinate independence of the path integrals. The rules are derived by using equations of motion and partial integration, while keeping track of certain minimal features originating in the unique definition of all singular integrals in $1-\epsilon$ dimensions. Our rules yield the same results as the much more cumbersome calculations in $1-\epsilon$ dimensions where the limit $\epsilon \to 0$ is taken at the end. They also agree with the rules found in an independent treatment on a finite time interval.

Introduction. – While quantum mechanical path integrals in curvilinear coordinates have been defined uniquely and independently of the choice of coordinates within the time-sliced formalism [1], a perturbative definition on a continuous time axis poses severe problems which have been solved only recently [2, 3]. To exhibit the origin of the difficulties, consider the associated partition function calculated for periodic paths on the imaginary-time axis τ:

$$Z = \int \mathcal{D}q(\tau)\, \sqrt{g(q)}\, e^{-\mathcal{A}[q]}, \tag{1}$$

where $\mathcal{A}[q]$ is the euclidean action with the general form

$$\mathcal{A}[q] = \int_0^\beta d\tau \left[\frac{1}{2}\, g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau) + V(q(\tau)) \right]. \tag{2}$$

The dots denote τ-derivatives, $g_{\mu\nu}(q)$ is a metric, and $g = \det g$ its determinant. The path integral is defined perturbatively as follows: The metric $g_{\mu\nu}(q)$ and the potential $V(q)$ are

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expanded around some point q_0^μ in powers of $\delta q^\mu \equiv q^\mu - q_0^\mu$. After this, the action $\mathcal{A}[q]$ is separated into a free part $\mathcal{A}_0[q_0;\delta q] \equiv \int_0^\beta d\tau \left[\frac{1}{2}g_{\mu\nu}(q_0)\partial_t \delta q^\mu \partial_t \delta q^\nu + \frac{1}{2}\omega^2 \delta q^\mu \delta q^\nu\right]$, and an interacting part $\mathcal{A}_{\rm int}[q_0;\delta q] \equiv \mathcal{A}[q] - \mathcal{A}_0[q_0;\delta q]$.

A first problem is encountered in the measure of functional integration in (1). Taking $\sqrt{g(q)}$ into the exponent and expanding in powers of δq, we define an effective action $\mathcal{A}_{\sqrt{g}} = -\frac{1}{2}\delta(0)\int_0^\beta d\tau \log[g(q_0+\delta q)/g(q_0)]$ which contains the infinite quantity $\delta(0)$, the δ-function at the origin. It is a formal representation of the inverse infinitesimal lattice spacing on the time axis, and is equal to the linearly divergent momentum integral $\int dp/(2\pi)$.

The second problem arises in the expansion of Z in powers of the interaction. Performing all Wick contractions, Z is expressed as a sum of loop diagrams. There are interaction terms involving $\delta\dot{q}^2\delta q^n$ which lead to Feynman integrals over products of distributions. The diagrams contain three types of lines representing the correlation functions

$$\begin{aligned} \Delta(\tau-\tau') &\equiv \langle \delta q(\tau)\delta q(\tau')\rangle = \quad , & (3)\\ \partial_\tau\Delta(\tau-\tau') &\equiv \langle \delta \dot{q}(\tau)\delta q(\tau')\rangle = \quad , & (4)\\ \partial_\tau\partial_{\tau'}\Delta(\tau-\tau') &\equiv \langle \delta \dot{q}(\tau)\delta \dot{q}(\tau')\rangle = \quad . & (5)\end{aligned}$$

The right-hand sides define the line symbols to be used in Feynman diagrams to follow below.

Explictly, the first correlation function reads

$$\Delta(\tau,\tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}. \tag{6}$$

The second correlation function has a discontinuity

$$\partial_\tau\Delta(\tau,\tau') = -\frac{1}{2}\epsilon(\tau-\tau')e^{-\omega|\tau-\tau'|}, \tag{7}$$

where

$$\epsilon(\tau-\tau') \equiv -1 + 2\int_{-\infty}^{\tau} d\tau''\delta(\tau''-\tau') \tag{8}$$

is a distribution which vanishes at the origin and is equal to ± 1 for positive and negative arguments, respectively. The third correlation function contains a δ-function:

$$\partial_\tau\partial_{\tau'}\Delta(\tau,\tau') = \delta(\tau-\tau') - \frac{\omega}{2}e^{-\omega|\tau-\tau'|}, \tag{9}$$

Mathematically, the temporal integrals over products of such distributions are undefined [4]. In this paper we specify these integrals by imposing the natural requirement of coordinate independence of the path integral (1). By the perturbative calculation up to three loops we show that this requirement alone can not fix uniquely values of all ambiguous integrals over products of distributions. However we define a simple consistent procedure for calculating singular Feynman integrals. All results obtained in this way ensure coordinate independence. They agree with what we have obtained in our previous work. In Ref. [2], we have shown that Feynman integrals in *momentum space* can be uniquely defined as $\epsilon \to 0$ -limits of $1-\epsilon$ -dimensional integrals via an analytic continuation à la 't Hooft and Veltman [5]. This definition makes path integrals coordinate-independent. In Ref. [3] we have given rules for calculating the same results directly from Feynman integrals in a $1-\epsilon$ -dimensional space.

The calculation procedure developed in this paper avoids the cumbersome evaluation of Feynman integrals in $1-\epsilon$ dimensions. In fact, it does not require specifying any regularization scheme. As a fundamental byproduct, it lays the foundation for a new extension of the theory of distributions, in which also integrals over products are defined, not only linear combinations.

Perturbation Expansion. – The relations between singular Feynman integrals will be derived form the requirement of coordinate independence of the exactly solvable path integral of a point particle of unit mass in a harmonic potential $\omega^2 x^2/2$, whose action is

$$\mathcal{A}_\omega = \frac{1}{2} \int d\tau \left[\dot{x}^2(\tau) + \omega^2 x^2(\tau)\right]. \tag{10}$$

For a large imaginary-time interval β, the partition function is given by the path integral

$$Z_\omega = \int \mathcal{D}x(\tau)\, e^{-\mathcal{A}_\omega[x]} = e^{-(1/2)\mathrm{Tr}\log(-\partial^2+\omega^2)} = e^{-\beta\omega/2}. \tag{11}$$

For simplicity here and in the following the target space is assumed to be one-dimensional. In a higher dimensional euclidean space the problem of coordinate independence of the path integral (11) was first examined in the papers [6] and [7] where different noncovariant quantum corrections to the classical action (10) were found. When this space being transformed holonomically to the curvilinear coordinates all covariant structures, such as the scalar curvature, remain, however, zero. Therefore the same problem can be reexamined directly in one dimension without loss of generality.

A coordinate transformation turns (11) into a path integral of the type (1) with a singular perturbation expansion. From our work in Refs. [2,3] we know that all terms in this expansion vanish in dimensional regularization, thus ensuring the coordinate independence of the perturbatively defined path integral. In this paper, we proceed in the opposite direction: we *require* the vanishing of all expansion terms to find the relations for integrals over products of distributions.

For simplicity we assume the coordinate transformation to preserve the symmetry $x \leftrightarrow -x$ of the initial oscillator, such that its power series expansion starts out like $x(\tau) = f(q(\tau)) = q - gq^3/3 + g^2 a q^5/5 - \cdots$, where g is a smallness parameter, and a some extra parameter. We shall see that the identities are independent of a, such that a will merely serve to check the calculations. The transformation changes the partition function (11) into

$$Z = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}_J[q]} e^{-\mathcal{A}[q]}, \tag{12}$$

where is $\mathcal{A}[q]$ is the transformed action, whereas $\mathcal{A}_J[q]$ an effective action coming from the Jacobian of the coordinate transformation:

$$\mathcal{A}_J[q] = -\delta(0) \int d\tau\, \log \frac{\delta f(q(\tau))}{\delta q(\tau)}. \tag{13}$$

The transformed action is decomposed into a free part

$$\mathcal{A}_\omega[q] = \frac{1}{2} \int d\tau [\dot{q}^2(\tau) + \omega^2 q^2(\tau)], \tag{14}$$

and an interacting part, which reads to second order in g:

$$\begin{aligned} \mathcal{A}_{\rm int}[q] &= \frac{1}{2} \int d\tau \left\{ -g \left[2\dot{q}^2(\tau) q^2(\tau) + \frac{2\omega^2}{3} q^4(\tau)\right] \right. \\ &+ \left. g^2 \left[(1+2a)\, \dot{q}^2(\tau) q^4(\tau) + \omega^2 \left(\frac{1}{9} + \frac{2a}{5}\right) q^6(\tau)\right] \right\}. \end{aligned} \tag{15}$$

To the same order in g, the Jacobian action (13) is

$$\mathcal{A}_J[q] = -\delta(0) \int d\tau \left[-g q^2(\tau) + g^2 \left(a - \frac{1}{2} \right) q^4(\tau) \right]. \tag{16}$$

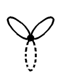
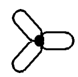
For $g = 0$, the transformed partition function (12) coincides, of course, with (11). When expanding Z of Eq. (12) in powers of g, we obtain Feynman integrals to each order in g, whose sum must vanish to ensure coordinate independence. By considering only connected Feynman diagrams, we study directly the ground state energy.

Ground State Energy. – Here the coordinate independence will be tested perturbatively up to three loops. The graphical expansion for the ground state energy has the following structure: To given order g^n, there exist Feynman diagrams with $L = n+1, n$, and $n-1$ number of loops coming from the interaction terms (15) and (16), respectively. The diagrams are composed of the three line types in (3)–(5), and new interaction vertices arising for each power of g. The diagrams coming from the Jacobian action (16) are easily recognized by accompanying factors $\delta^n(0)$.

To first order in g, there exists only three diagrams, two originated from the interaction (15), and one from the Jacobian action (16):

$-g$  $-g\,\omega^2$  $+g\,\delta(0)$  . (17)

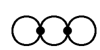
To order g^2, we distinguish several contributions. First there are two three-loop local diagrams coming from the interaction (15), and one two-loop local diagram from the Jacobian action (16):

$g^2 \Big[\, 3\left(\frac{1}{2}+a\right)$  $+\, 15\omega^2\left(\frac{1}{18}+\frac{a}{5}\right)$  $-\, 3\left(a-\frac{1}{2}\right)\delta(0)$  $\Big].$ (18)

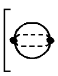
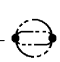
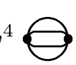
We call a diagram *local* if it involves no temporal integral. The Jacobian action (16) contributes further the nonlocal diagrams:

$-\frac{g^2}{2!}\Big\{ 2\delta^2(0)$  $-\, 4\delta(0)\big[$  $+$  $+\, 2\,\omega^4$  $\big]\Big\}.$ (19)

The remaining diagrams come from the interaction (15) only. They are either of the three-bubble type, or of the watermelon type, each with all possible combinations of the three line types (3)–(5): The sum of all three-bubbles diagrams is

$-\frac{g^2}{2!}\big[\; 4$  $+\; 2$  $+\; 2$ 
$+\; 8\,\omega^2$  $+\, 8\omega^2$  $+\, 8\omega^4$  $\big].$ (20)

The watermelon-like diagrams contribute

$-\frac{g^2}{2!}\, 4 \Big[$  $+\, 4$  $+$  $+\, 4\omega^2$  $+\frac{2}{3}\omega^4$  $\Big].$ (21)

Since the equal-time expectation value $\langle \dot{q}(\tau)\, q(\tau) \rangle$ vanishes according to Eq. (7) there are, in addition, a number of trivially vanishing diagrams, which have been omitted.

In our previous work [2, 3], all integrals were calculated in $D = 1 - \varepsilon$ dimensions, taking the limit $\varepsilon \to 0$ at the end. In this way we confirmed that the sums of all Feynman diagrams contributing to each order in g vanish. Here we proceed in the opposite direction and derive the rules for integrating products of distributions from the vanishing of the sums.

Imposing Coordinate Independence. – In a first step we simplify the above perturbation expansion of the ground state energy using the inhomogeneous field equation satisfied by the correlation function (6):

$$\ddot{\Delta}(\tau) = -\int đk\, \frac{k^2}{k^2+\omega^2} e^{ik\tau} = -\,\delta(\tau) + \omega^2 \Delta(\tau)\,, \tag{22}$$

and the following well-defined integrals for two correlation functions:

$$\int d\tau \left[\dot{\Delta}^2(\tau) + \omega^2 \Delta^2(\tau)\right] = \Delta(0), \tag{23}$$

as well as the integrals containing four correlation functions:

$$\int d\tau\, \Delta^4(\tau) = \frac{1}{4\omega^2}\,\Delta^3(0), \tag{24}$$

$$\int d\tau\, \dot{\Delta}^2(\tau)\Delta^2(\tau) = \frac{1}{4}\,\Delta^3(0), \tag{25}$$

$$\int d\tau\, \dot{\Delta}^4(\tau) = \frac{1}{4}\,\omega^2\,\Delta^3(0). \tag{26}$$

These integrals are obtained directly by substituting the explicit representations (6) and (7) into (23), (24), (25) and (26), respectively. Also we shall use the singular integral obtained by the same direct substitution of two correlation functions:

$$\int d\tau \left[\ddot{\Delta}^2(\tau) + 2\omega^2\dot{\Delta}^2(\tau) + \omega^4\Delta^2(\tau)\right] = \int d\tau\, \delta^2(\tau)\,, \tag{27}$$

where the last integral containing the square of the δ-function will be specified later by the requirement of coordinate independence.

Consider now the perturbation expansion of the ground state energy. To first order in g, the sum of Feynman diagrams (17) must vanish:

 $+\ \omega^2$  $-\ \delta(0)$  $= 0.$ (28)

The analytic form of this relation is

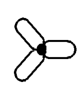
$$\left[-\ddot{\Delta}(0) + \omega^2\Delta(0) - \delta(0)\right]\,\Delta(0) = 0, \tag{29}$$

the zero on the right-hand side being a direct consequence of the equation of motion (22) for the correlation function at origin.

To order g^2, the same equation reduces the sum of all local diagrams in (18) to a finite result plus a term proportional to $\delta(0)$:

$$\left[-3\left(\frac{1}{2}+a\right)\ddot{\Delta}(0) + 15\left(\frac{1}{18}+\frac{a}{5}\right)\omega^2\Delta(0) - 3\left(a-\frac{1}{2}\right)\delta(0)\right]\Delta^2(0) = \left[3\delta(0) - \frac{2}{3}\,\omega^2\Delta(0)\right]\Delta^2(0).$$

Representing right-hand side diagrammatically, we obtain the identity

$\Sigma\,(18) = 3\delta(0)$  $-\ \frac{2}{3}\,\omega^2$ , (30)

where $\Sigma\,(18)$ denotes the sum of all diagrams in Eq. (18). Using (23) together with the field equation (22), we reduce the sum (19) of all one and two-loop bubbles diagrams to terms involving $\delta(0)$ and $\delta^2(0)$:

$$-\frac{2}{2!}\left\{\delta^2(0)\int d\tau\,\Delta^2(\tau) - 2\delta(0)\int d\tau\,\left[\Delta(0)\dot{\Delta}^2(\tau) - \ddot{\Delta}(0)\Delta^2(\tau) + 2\omega^2\Delta(0)\Delta^2(\tau)\right]\right\}$$
$$= 2\delta(0)\,\Delta^2(0) + \delta^2(0)\int d\tau\Delta^2(\tau). \tag{31}$$

Hence we find the diagrammatic identity

$\Sigma\,(19) = 2\delta(0)$  $+\ \delta^2(0)$  . (32)

Now, the terms accompanying $\delta^2(0)$ turn out to cancel similar terms coming from the sum of all three-loop bubbles diagrams in (20). In fact, the relations (23) and (27) lead to

$$-\frac{2}{2!}\int d\tau\,\left[-2\Delta(0)\ddot{\Delta}(0)\dot{\Delta}^2(\tau) + \Delta^2(0)\ddot{\Delta}^2(\tau) + \ddot{\Delta}^2(0)\Delta^2(\tau) + 4\omega^2\Delta^2(0)\dot{\Delta}^2(\tau)\right.$$
$$\left.-4\omega^2\Delta(0)\ddot{\Delta}(0)\Delta^2(\tau) + 4\omega^4\Delta^2(0)\Delta^2(\tau)\right] = -\left[\int d\tau\,\delta^2(\tau) + 2\delta(0)\right]\Delta^2(0) - \delta^2(0)\int d\tau\,\Delta^2(\tau).$$

Thus we find the diagrammatic identity for all bubbles diagrams

$\Sigma\,(19) + \Sigma\,(20) = -\int d\tau\,\delta^2(\tau)$  . (33)

The first two watermelon diagrams in Eq. (21) correspond to the integrals

$$I_1 = \int d\tau\,\ddot{\Delta}^2(\tau)\Delta^2(\tau)\,, \tag{34}$$
$$I_2 = \int d\tau\,\ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau), \tag{35}$$

whose evaluation is subtle. Consider first the integral (34) which contains a square of a δ-function. We separate this out by writing

$$I_1 = \int d\tau\,\ddot{\Delta}^2(\tau)\Delta^2(\tau) = I_1^{\rm div} + I_1^R, \tag{36}$$

with a divergent and a regular part

$$I_1^{\rm div} = \Delta^2(0)\int d\tau\,\delta^2(\tau)\,, \qquad I_1^R = \int d\tau\,\Delta^2(\tau)\left[\ddot{\Delta}^2(\tau) - \delta^2(\tau)\right]. \tag{37}$$

All other watermelon diagrams (21) lead to the well-defined integrals (26), (25), and (24), respectively. Substituting these and (35), (36) into (21) yields the sum of all watermelon diagrams

$$-\frac{4}{2!}\int d\tau\,\left[\Delta^2(\tau)\ddot{\Delta}^2(\tau) + 4\Delta(\tau)\dot{\Delta}^2(\tau)\ddot{\Delta}(\tau) + \dot{\Delta}^4(\tau) + 4\omega^2\Delta^2(\tau)\dot{\Delta}^2(\tau) + \frac{2}{3}\omega^4\Delta^4(\tau)\right]$$
$$= -\,2\Delta^2(0)\int d\tau\,\delta^2(\tau) - 2\left(I_1^R + 4I_2\right) - \frac{17}{6}\,\omega^2\Delta^3(0)\,. \tag{38}$$

Adding these to (30), (33), we obtain the sum of all second-order connected diagrams

$$\Sigma\,(\text{all}) = 3\left[\delta(0) - \int d\tau\,\delta^2(\tau)\right]\Delta^2(0) - 2\,\left(I_1^R + 4I_2\right) - \frac{7}{2}\,\omega^2\Delta^3(0)\,, \tag{39}$$

where the integrals I_1^R and I_2 are undefined so far. This sum has to vanish to guarantee coordinate independence. We therfore equate to zero both the singular and finite contributions in Eq. (39). The first yields the rule for the product of two δ-functions

$$\delta^2(\tau) = \delta(0)\,\delta(\tau)\,. \tag{40}$$

This equality should of course be understood in the distributional sense, it holds after multiplying it with an arbitrary test function and integrating over τ. As it was shown in our previous paper [8], the rule (40) guarantees the cancellation of all short-distance singularities to any order of the perturbation theory. This happens independently of the value of $\delta(0)$, making a regularization superfluous. There is perfect cancellation of all powers of $\delta(0)$ arising from the expansion of the Jacobian action, which is the fundamental reason why the heuristic *Veltman rule* of setting $\delta(0) = 0$ is applicable everywhere without problems.

The vanishing of the regular parts of (39) requires the integrals (35) and (36) to satisfy

$$I_1^R + 4I_2 = -\frac{7}{4}\,\omega^2\Delta^3(0) = -\frac{7}{32\omega}\,. \tag{41}$$

At this point we run into two difficulties. First, this single equation (41) for the two undefined integrals I_1^R and I_2 is insufficient to specify both integrals, so that the requirement of reparametrization invariance alone is not enough to fix all ambiguous temporal integrals over products of distributions. Second, and more seriously, Eq. (41) leads to conflicts with standard integration rules based on the use of partial integration and equation of motion. Let us apply these rules to the calculation of the integrals I_1^R and I_2 in different orders. Inserting the equation of motion (22) into the finite part of the integral (36) and making use of the regular integral (24), we find immediately

$$\begin{aligned} I_1^R &= \int d\tau\,\Delta^2(\tau)\left[\ddot{\Delta}^2(\tau) - \delta^2(\tau)\right] \\ &= -2\omega^2\,\Delta^3(0) + \omega^4\int d\tau\,\Delta^4(\tau) = -\frac{7}{4}\omega^2\Delta^3(0) = -\frac{7}{32\omega}\,. \end{aligned} \tag{42}$$

The same substitution of the equation of motion (22) into the other ambiguous integral I_2 of (35) leads, after performing the the regular integral (25), to

$$\begin{aligned} I_2 &= -\int d\tau\,\dot{\Delta}^2(\tau)\,\Delta(\tau)\,\delta(\tau) + \omega^2\int d\tau\,\dot{\Delta}^2(\tau)\,\Delta^2(\tau) \\ &= -\frac{1}{8\omega}\int d\tau\,\epsilon^2(\tau)\,\delta(\tau) + \frac{1}{4}\omega^2\Delta^3(0) = \frac{1}{8\omega}\left(-I + \frac{1}{4}\right)\,, \end{aligned} \tag{43}$$

where I denotes the undefined integral over a product of distributions

$$I = \int d\tau\,\epsilon^2(\tau)\delta(\tau)\,. \tag{44}$$

This integral can apparently be fixed by applying partial integration to the integral (35) which reduces it to the completely regular form (26):

$$I_2 = \frac{1}{3}\int d\tau\,\Delta(\tau)\frac{d}{d\tau}\left[\dot{\Delta}^3(\tau)\right] = -\frac{1}{3}\int d\tau\,\dot{\Delta}^4(\tau) = -\frac{1}{12}\omega^2\Delta^3(0) = -\frac{1}{96\omega}. \tag{45}$$

There are no boundary terms due to the exponential vanishing at infinity of all functions involved. From (43) and (45) we conclude that $I = 1/3$.

At this point we observe a conflict. The results (45) and (42) do not obey the equation (41), such that they are incompatible with the necessary coordinate independence of the path integral. This was the reason to add a noncovariant quantum correction term $\Delta V = -g^2(q^2/6)$ to the classical action (10) in the previous paper [7]. For perturbative calculation on a finite-time interval it was also done in [9].

From the perspective of our previous papers [2,3] where all integrals were defined in $d = 1-\epsilon$ dimensions and continued to $\epsilon \to 0$ at the end, the above-observed inconsistency is obvious: Arbitrary application of partial integration and equation of motion to one-dimensional integrals is forbidden whenever several dots can correspond to different contractions of partial derivatives ∂_α, $\partial_\beta, \ldots$, from which they arise in the limit $d \to 1$. The different contractions may lead to different integrals. In the pure one-dimensional calculation of the integrals I_1^R and I_2 this ambiguity can be accounted for by using partial integration and equation of motion (22) only according to the following integration rules:

1. We perform a partial integration which allows us to apply subsequently the equation of motion (22).

2. If the equation of motion (22) leads to integrals of the type (44), they must be performed using the naively the Dirac rule for the δ-function and the property $\epsilon(0) = 0$.

3. The above procedure leaves in general singular integrals, which must be treated once more with the same rules.

Let us show that calculating the integrals I_1^R and I_2 with these rules is consistent with the coordinate independence condition (41). In the integral I_2 of (35) we first apply partial integration to find

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int d\tau\, \Delta(\tau)\, \dot{\Delta}(\tau) \frac{d}{d\tau}\left[\dot{\Delta}^2(\tau)\right] \\
&= -\frac{1}{2}\int d\tau\, \dot{\Delta}^4(\tau) - \frac{1}{2}\int d\tau\, \Delta(\tau)\, \dot{\Delta}^2(\tau)\, \ddot{\Delta}(\tau)\,, \qquad (46)
\end{aligned}
$$

with no contributions from the boundary terms. Note that the partial integration (45) is forbidden since it does not allow for a subsequent application of the equation of motion (22). On the right-hand side of (46) it can be applied. This leads to a combination of two regular integrals (25) and (26) and the singular integral I, which we evaluate with the naive Dirac rule to $I = 0$, resulting in

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int d\tau\, \dot{\Delta}^4(\tau) + \frac{1}{2}\int d\tau\, \dot{\Delta}^2(\tau)\, \Delta(\tau)\, \delta(\tau) - \frac{1}{2}\omega^2 \int d\tau\, \dot{\Delta}^2(\tau)\, \Delta^2(\tau) \\
&= \frac{1}{16\omega}\, I - \frac{1}{4}\omega^2 \Delta^3(0) = -\frac{1}{32\omega}\,. \qquad (47)
\end{aligned}
$$

If we calculate the finite part I_1^R of the integral (36) with the new rules we obtain a result different from (42). Integrating the first term in brackets by parts and using the equation of motion (22), we obtain

$$
\begin{aligned}
I_1^R &= \int d\tau \Delta^2(\tau)\left[\ddot{\Delta}^2(\tau) - \delta^2(\tau)\right] \\
&= \int d\tau \left[-\,\dddot{\Delta}\,(\tau)\, \dot{\Delta}(\tau)\, \Delta^2(\tau) - 2\ddot{\Delta}(\tau)\, \dot{\Delta}^2(\tau)\, \Delta(\tau) - \Delta^2(\tau)\, \delta^2(\tau)\right] \\
&= \int d\tau \left[\dot{\Delta}(\tau)\, \Delta^2(\tau)\, \dot{\delta}(\tau) - \Delta^2(\tau)\, \delta^2(\tau)\right] - 2I_2 - \omega^2 \int d\tau \dot{\Delta}^2(\tau)\, \Delta^2(\tau)\,. \qquad (48)
\end{aligned}
$$

The last two terms are already known, while the remaining singular integral in brackets must be subjected once more to the same treatment. It is integrated by parts so that the equation of motion (22) can be applied to obtain

$$\int d\tau \left[\dot\Delta(\tau)\,\Delta^2(\tau)\,\dot\delta(\tau) - \Delta^2(\tau)\,\delta^2(\tau)\right]$$
$$= -\int d\tau \left[\ddot\Delta(\tau)\,\Delta^2(\tau) + 2\dot\Delta^2(\tau)\,\Delta(\tau)\right]\delta(\tau) - \int d\tau \Delta^2(\tau)\,\delta^2(\tau) = -\omega^2\,\Delta^3(0) - \frac{1}{4\omega}\,I\,. \quad (49)$$

Inserting this into Eq. (48) yields

$$I_1^R \;=\; \int d\tau \Delta^2(\tau)\left[\ddot\Delta^2(\tau) - \delta^2(\tau)\right] = -2I_2 - \frac{5}{4}\omega^2\,\Delta^3(0) - \frac{1}{4\omega}\,I = -\frac{3}{32\omega}\,, \quad (50)$$

the right-hand side following for $I = 0$.

We see now that the integrals (47) and (50) calculated with the new rules obey the equation (41) which guarantees coordinate independence of the path integral.

The applicability of the rules (1-3) follows immediately from the previously established dimensional continuation [2, 3]. It avoids completely the cumbersome calculations in $1-\varepsilon$-dimension with the subsequent limit $\varepsilon \to 0$. Only some intermediate steps of the derivation require keeping track of the d-dimensional origin of the rules. For this, we continue the imaginary time coordinate τ to a d-dimensional spacetime vector $\tau \to x^\mu = (\tau, x^1, \ldots, x^{d-1})$. and note that the equation of motion (22) becomes a scalar field equation of the Klein-Gordon type

$$\left(-\partial_\alpha^2 + \omega^2\right)\Delta(x) = \delta^{(d)}(x)\,. \quad (51)$$

In d dimensions, the relevant second-order diagrams are obtained by decomposing the harmonic expectation value

$$\int d^dx \, < q_\alpha^2(x)\, q^2(x)\, q_\beta^2(0)\, q^2(0) > \quad (52)$$

into a sum of products of four two-point correlation functions according to the Wick rule. The fields $q_\alpha(x)$ are the d-dimensional extensions $q_\alpha(x) \equiv \partial_\alpha q(x)$ of $\dot q(x)$. Now the d-dimensional integrals, corresponding to the integrals (34) and (35), are defined uniquely by the contractions

$$\begin{aligned} I_1^d &= \int d^dx \, < q_\alpha(x) q_\alpha(x) q(x) q(x) q_\beta(0) q_\beta(0) q(0) q(0) > \\ &= \int d^dx \, \Delta^2(x)\,\Delta_{\alpha\beta}^2(x)\,, \end{aligned} \quad (53)$$

$$\begin{aligned} I_2^d &= \int d^dx \, < q_\alpha(x) q_\alpha(x) q(x) q(x) q_\beta(0) q_\beta(0) q(0) q(0) > \\ &= \int d^dx \, \Delta(x)\,\Delta_\alpha(x)\,\Delta_\beta(x)\,\Delta_{\alpha\beta}(x)\,. \end{aligned} \quad (54)$$

The different derivatives $\partial_\alpha\partial_\beta$ acting on $\Delta(x)$ prevent us from applying the field equation (51). This obstacle was hidden in the one-dimensional formulation. It can be overcome by a partial integration. Starting with I_2^d, we obtain

$$I_2^d \;=\; -\frac{1}{2}\int d^dx \, \Delta_\beta^2(x)\left[\Delta_\alpha^2(x) + \Delta(x)\,\Delta_{\alpha\alpha}(x)\right]\,, \quad (55)$$

Treating I_1^d likewise we find

$$I_1^d = -2I_2^d + \int d^dx\, \Delta^2(x)\, \Delta^2_{\alpha\alpha}(x) + 2\int d^dx\, \Delta(x)\, \Delta^2_\beta(x)\, \Delta_{\alpha\alpha}(x)\,. \tag{56}$$

In the second equation we have used the fact that $\partial_\alpha \Delta_{\alpha\beta} = \partial_\beta \Delta_{\lambda\lambda}$. The right-hand sides of (55) and (56) contain now the contracted derivatives ∂_α^2 such that we can apply the field equation (51). This mechanism works to all orders in the perturbation expansion which is the reason for the applicability of the rules (1) and (2) which led to the results (47) and (50) ensuring coordinate independence.

The rule (3) is a consequence of regularized equation in $d = 1 - \varepsilon$ dimension

$$\Delta(x)\, \Delta^2_\alpha(x)\, \delta(x) \simeq |x|^{2\varepsilon}\, \delta(x) = 0\,, \tag{57}$$

which makes the singular product of distributions in Eq. (44) vanishes before taking $\varepsilon \to 0$-limit. This rule is in agreement with the earlier result derived for finite time intervals in Ref. [8].

Let us illustrate how different contractions of partial derivatives ∂_α, $\partial_\beta, \ldots$, may lead to different integrals in the $d \to 1$-limit. The simplest example is the anomalous integral

$$I^d_{\rm an} = \int d^dx\, \Delta^2(x)\, \Delta^2_{\alpha\beta}(x) - \int d^dx\, \Delta^2(x)\, \Delta^2_{\alpha\alpha}(x)\,, \tag{58}$$

where the integrals on the right-hand side are indistinguishable in $d = 1$ dimension. However, it follows immediately from Eqs. (55) and (56) that

$$\begin{aligned} I^d_{\rm an} &= \int d^dx\, \Delta^2_\alpha(x)\, \Delta^2_\beta(x) + 3\int d^dx\, \Delta(x)\, \Delta^2_\beta(x)\, \Delta_{\alpha\alpha}(x) \\ &= \int d^dx\, \Delta^2_\alpha(x)\, \Delta^2_\beta(x) + 3\omega^2 \int d^dx\, \Delta^2(x)\, \Delta^2_\beta(x) \\ &- 3\int d^dx\, \Delta(x)\, \Delta^2_\beta(x) \delta(x) \underset{d\to 1}{\longrightarrow} \frac{1}{8\omega}. \end{aligned} \tag{59}$$

The dimensional continuation of all other second-order diagrams does not change the results of one-dimensional calculation. Only the second diagram in Eq. (20) requires taking more care. However, the corresponding d-dimensional integral has the property

$$\Delta^2(0) \int d^dx\, \Delta^2_{\alpha\beta}(x) = \Delta^2(0) \int d^dx\, \Delta^2_{\alpha\alpha}(x)\,, \tag{60}$$

which allows us to apply the equation of motion (51) directly as in one dimension. Thus, by keeping only track of a few essential properties of the theory in d dimensions we indeed obtain a simple consistent procedure for calculating singular Feynman integrals. All results obtained in this way ensure coordinate independence. Our procedure gives us unique rules telling us where we are allowed to apply partial integration and the equation of motion in one-dimensional expressions. Ultimately, all integrals are brought to a regular form, which can be continued back to one time dimension for a direct evaluation. This procedure is obviously much simpler than the previous explicit calculations in d-dimension with the limit $d \to 1$ taken at the end.

The above rules for integrating products of distributions are in complete agreement with the rules derived earlier in a different way for path integrals on a finite time interval where the infrared regulator $\omega^2 q^2$ was not needed in the free part of the action $\mathcal{A}_0$.

Let us briefly discuss the alternative possibility of giving up partial integration completely in ambiguous integrals containing ϵ- and δ-function, or their time derivatives, which makes unnecessary to satisfy Eq. (45). This yields a freedom in the definition of integral over product of distribution (44) which can be used to fix $I = 1/4$ from the requirement of coordinate independence [10]. Indeed, this value of I makes the integral (43) equal to $I_2 = 0$ such that (41) is satisfied and coordinate independence ensured. In contrast, giving up partial integration, the authors of Refs. [6, 11] have assumed the vanishing $\epsilon^2(\tau)$ at $\tau = 0$ so that the integral I should vanish as well: $I = 0$. Then Eq. (43) yields $I_2 = 1/32\omega$ which together with (42) does not obey the coordinate independence condition (41), making yet an another noncovariant quantum correction $\Delta V = g^2(q^2/2)$ necessary in the action, which we reject since it contradicts Feynman's original rules of path integration. We do not consider giving up partial integration as an attractive option since it is an important tool for calculating higher-loop diagrams.

Summary. – We have defined singular Feynman integrals in perturbative calculation of zero-temperature path integrals in such a way that coordinate independence is guaranteed. To second-order in perturbation theory, one encounters two singular integrals containing products of distributions and finds one relation between them due to coordinate independence. The evaluation of the individual integrals was made unique by analytic regularization [2,3]. In this paper we have shown that there are simple rules for obtaining the same results without the analytic continuation of Feynman integrals to d dimensions. We have merely kept track of the different contractions of the derivatives and performed partial integrations until one obtains Laplace operators which allow us to use the d-dimensional field equation which removes one singularity. If repeated recursively, this procedure leads to regular integrals. There is no need to specify a regularization scheme, and calculations are much simpler than the previous ones in d-dimension with the limit $d \to 1$ taken at the end. Our results are agreement with the earlier result derived for finite time intervals in Ref. [8].

Just as in the time-sliced definition of path integrals in curved space in Ref. [1], there is absolutely no need for extra compensating potential terms found necessary in the treatments in Refs. [6, 10, 7].

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the extra terms in Appendix A of the first paper required by the noncovariant regularization of these authors.